18.06 Professor Edelman Quiz 1 October 3, 2012

Grading

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Your PRINTED name is: ________________________

Please circle your recitation:

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1 (22 pts.)

Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 3 & 4 \end{pmatrix}$ and $M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 3 & 4 \end{pmatrix}$.

a) (5 pts.) Which are the pivot columns and which are the free columns of A ?

We subtract twice the first row from the second and three times the first row from the third to find the row echelon form:

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We see that the first column is free and the second and third columns are pivots.

b) (5 pts.) Which are the pivot columns and which are the free columns of M ?

We subtract twice the first row from the second and three times the first row from the third to find the row echelon form:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We see that all three columns are pivot columns and thus there are no free columns.

c) (6 pts.) For which $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ are there solutions to $Ax = b$? For those b, write down the complete solution.

We form the augmented matrix:

$$\begin{pmatrix} 0 & 1 & 1 & b_1 \\ 0 & 2 & 2 & b_2 \\ 0 & 3 & 4 & b_3 \end{pmatrix}$$

As before, we subtract twice the first row from the second and three times the first row from the third:

$$\begin{pmatrix} 0 & 1 & 1 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 1 & b_3 - 3b_1 \end{pmatrix}.$$

We now find the reduced echelon form by subtracting the third row from the first:

$$\begin{pmatrix} 0 & 1 & 0 & 4b_1 - b_3 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 1 & b_3 - 3b_1 \end{pmatrix}.$$

The second row gives the equation $0 = b_2 - 2b_1$ so for $Ax = b$ to have solutions, we must have $b_2 = 2b_1$.

The third row gives $x_3 = b_3 - 3b_1$. The first row gives $x_2 = 4b_1 - b_3$.

The nullspace is 1-dimensional and by inspection we see that it contains $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, so the general solution to $Ax = b$ is

$$x = \begin{pmatrix} c \\ 4b_1 - b_3 \\ b_3 - 3b_1 \end{pmatrix},$$

where c may be any real number.

d) (6 pts.) For which $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ are there solutions to $Mx = b$? For those b, write down the complete solution.

Since M is square has no free columns, for any b there will be a solution to $Mx = b$. We form the augmented matrix:

$$\begin{pmatrix} 0 & 1 & 1 & b_1 \\ 1 & 2 & 2 & b_2 \\ 0 & 3 & 4 & b_3 \end{pmatrix}$$

As before, we subtract twice the first row from the second and three times the first row from the third:

$$\begin{pmatrix} 0 & 1 & 1 & b_1 \\ 1 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 1 & b_3 - 3b_1 \end{pmatrix}.$$

We now find the reduced echelon form by subtracting the third row from the first:

$$\begin{pmatrix} 0 & 1 & 0 & 4b_1 - b_3 \\ 1 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 1 & b_3 - 3b_1 \end{pmatrix}.$$

The third equation gives $x_3 = b_3 - 3b_1$, the first equation gives $x_1 = 4b_1 - b_3$, and the second equation gives $x_2 = b_2 - 2b_1$.

2 (24 pts.)

Consider the vector space of polynomials of the form $p(x) = ax^3 + bx^2 + cx + d$. Are the following subspaces? Explain briefly in a way that we are sure you understand subspaces.Note: We have written down the answers in detail to be pedagogical, but you didn't need to write so much.

a) (6 pts.) Those $p(x)$ for which $p(1) = 0$.

This is a subspace, because any linear combination of polynomials of degree at most 3 with a root at 1 is still a polynomial with degree at most 3 and a root at 1 : let $p(x) = fp_1(x)+gp_2(x)$ where f, g are any real number, then $p(1) = fp_1(1) + gp_2(1) = f \cdot 0 + g \cdot 0 = 0$, as required.

b) (6 pts.) Those $p(x)$ for which $p(0) = 1$.

This is not a vector space. One of many reasons is I can add two polynomials that have value 1 at 0. Then they will have value 2 at 0. Hence we have exhibited a linear combination of polynomials that are in the set which is not itself in the set. Hence this cannot be a subspace.

c) (6 pts.) Those $p(x)$ for which $a + b = c + d$.

This is a subspace. Consider two polynomials in the set, say $p_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i$, $i = 1, 2$. Then, since they are in the set, we know $a_i + b_i = c_i + d_i$ for $i = 1, 2$. Now take any linear combination of those polynomials, say $p(x) = fp_1(x) + gp_2(x)$. Writing $p(x)$ all out and rearranging terms, we get $p(x) = ax^3 + bx^2 + cx + d$, where $a = a_1 + a_2$, $b = b_1 + b_2$, etc. Clearly now, using $a_i + b_i = c_i + d_i$ for $i = 1, 2$. we have $a + b = c + d$. Hence $p(x)$, or in fact any linear combination of members of the set, is still in the set: we have a subspace.

d) (6 pts.) Those $p(x)$ for which $a^2 + b^2 = c^2 + d^2$.

This is not a subspace. You can guess from what we wrote down for c) that here, things might go wrong. For example, take $p_1(x) = x^3 + x$ and $p_2(x) = -x^3 + x$. Both of them are in the set, but if you add them up you get $p(x) = p_1(x) + p_2(x) = 2x$, which is not in the set.

3 (27 pts.)

a) (9 pts.) Find an LU decomposition of the matrix $A = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$, where we assume $a \neq 0$. L is unit lower triangular (1's on the diagonal) and U is upper triangular.

Use elimination. The upper-left entry a is our pivot, and we want to eliminate the c just below it, so subtract c/a times row 1 from row 2, to get

$$U = \begin{pmatrix} a & b \\ 0 & -bc/a \end{pmatrix}.$$

Our elimination matrix was

$$E_{21} = \begin{pmatrix} 1 & 0 \\ -c/a & 1 \end{pmatrix},$$

so

$$L = {E_{21}}^{-1} = \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix}.$$

We have $A = LU$ for the above L and U.

b) (9 pts.) Find a "PU" decomposition of the matrix $A = \begin{pmatrix} 0 & a & b \\ c & d & e \\ 0 & 0 & f \end{pmatrix}$, where P is a permutation matrix, and U is upper triangular.

The matrix A fails to be upper triangular because of the c in the first column, so we swap rows 1 and 2. Then $A = PU$, where

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} c & d & e \\ 0 & a & b \\ 0 & 0 & f \end{pmatrix}.$$

(Note that in this case $PA = U$ also, because $P = P^{-1}$. However, the problem asked for an $A = PU$ decomposition, so just writing $PA = U$ is not enough.)

c) (9 pts.) Find an "X'X" decomposition of the matrix $A = \begin{pmatrix} a^2+b^2+c^2 & ad+be+cf \\ ad+be+cf & d^2+e^2+f^2 \end{pmatrix}$. The matrix X that you need to find satisfies $A = X^T X$, and need not be a square matrix.

By inspection, $A = X^T X$ where X is the 3×2 matrix

$$X = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}.$$

Many other matrices X would work, but why make things unnecessarily complicated?

4 (27 pts.)

Either construct a matrix A or argue that it is impossible, where the nullspace of A is exactly the multiples of $(1, 1, 1, 1)$ and the dimensions (number of rows, number of columns) of A are

a) (9 pts.) 2×4

This is impossible. Indeed, a 2 by 4 matrix has rank at most 2 (the rank is the dimension of the row space). We know that $dimN(A) + dimC(A) = 4$, therefore, the dimension of the null space is at least 2, but the problem requires the null space to have dimension 1.

b) (9 pts.) 3×4

This one is possible. Since the null space has dimension 1, the rank has to be 3. If we find a rank 3 matrix having $(1, 1, 1, 1)$ in its null space then it's going to answer the problem. The following matrix works :

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

The ranks is 3 because there are 3 pivot columns and $(1, 1, 1, 1)$ is in the null space because the sum of the coefficients in each row is 0.

c) (9 pts.) 4×4

Here we need a rank 3 matrix with $(1, 1, 1, 1)$ in its null space. One possible answer is :

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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